12.4 Quasiconcavity and Quasiconvexity

In Sec. 11.5 it was shown that, for a problem of free extremum, a knowledge of the concavity or convexity of the objective function obviates the need to check the second-order condition. In the context of constrained optimization, it is again possible to dispense with the second-order condition if the surface or hypersurface has the appropriate type of configuration. But this time the desired configuration is quasiconcavity (rather than concavity) for a maximum, and quasiconvexity (rather than convexity) for a minimum. As we shall demonstrate, quasiconcavity (quasiconvexity) is a weaker condition than concavity (convexity). This is only to be expected, since the second-order sufficient condition to be dispensed with is also weaker for the constrained optimization problem (d^2z definite in sign only for those dx_i satisfying $dg = 0$) than for the free one (d^2z definite in sign for *all* dx_i).

Geometric Characterization

Quasiconcavity and quasiconvexity, like concavity and convexity, can be either strict or nonstrict. We shall first present the geometric characterization of these concepts:

> Let u and v be any two distinct points in the domain (a convex set) of a function f, and let line segment uv in the domain give rise to arc MN on the graph of the function, such that point N is higher than or equal in height to point M. Then function f is said to be *quasiconcave* (*quasiconvex*) if all points on arc MN other than M and N are higher than or equal in height to point M (lower than or equal in height to point N). The function f is said to be *strictly quasiconcave* (*strictly quasiconvex*) if all the points on arc MN other than M and N are strictly higher than point M (strictly lower than point N).

It should be clear from this that any strictly quasiconcave (strictly quasiconvex) function is quasiconcave (quasiconvex), but the converse is not true.

For a better grasp, let us examine the illustrations in Fig. 12.3, all drawn for the one-variable case. In Fig.12.3*a*, line segment uv in the domain gives rise to arc MN on the curve such that N is higher than M. Since all the points between M and N on the said arc are strictly higher than M, this particular arc satisfies the condition for strict quasiconcavity. For the curve to qualify as strictly quasiconcave, however, *all* possible (u, v) pairs must have arcs that satisfy the same condition. This is indeed the case for the function in Fig. 12.3*a*. Note that this function also satisfies the condition for (nonstrict) quasiconcavity. But it fails the condition for quasiconvexity, because some points on arc MN are higher than N, which is forbidden for a quasiconvex function. The function in Fig. 12.3*b* has the opposite configuration. All the points on arc $M'N'$ are lower than N', the higher of the two ends, and the same is true of all arcs that can be drawn. Thus the function in Fig. 12.3*b* is strictly

FIGURE 12.3

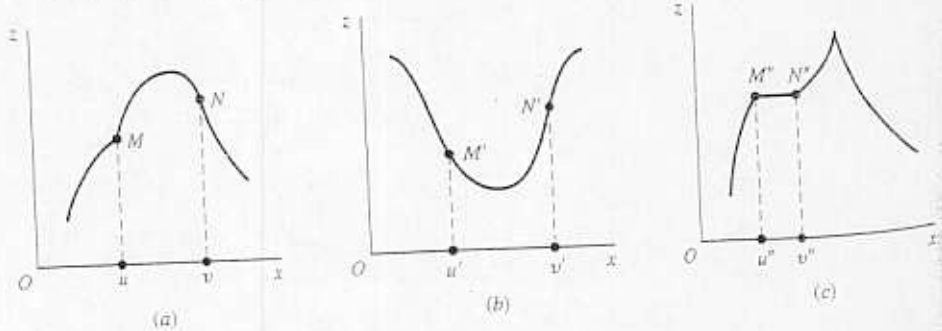
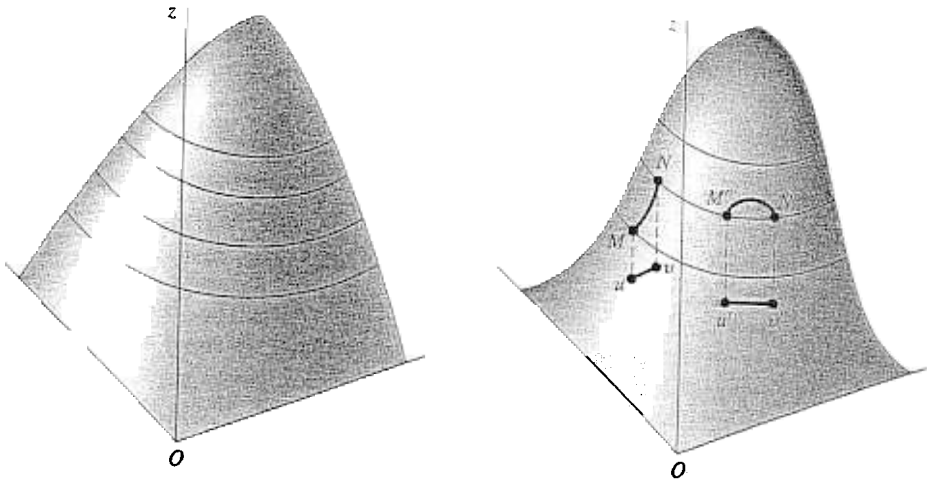


FIGURE 12.4



quasiconvex. As you can verify, it also satisfies the condition for (nonstrict) quasiconvexity, but fails the condition for quasiconcavity. What distinguishes Fig. 12.3*c* is the presence of a horizontal line segment $M''N''$, where all the points have the same height. As a result, that line segment—and hence the entire curve—can only meet the condition for quasiconcavity, but not strict quasiconcavity.

Generally speaking, a quasiconcave function that is not also concave has a graph roughly shaped like a bell, or a portion thereof, and a quasiconvex function has a graph shaped like an inverted bell, or a portion thereof. On the bell, it is admissible (though not required) to have both concave and convex segments. This more permissive nature of the characterization makes quasiconcavity (quasiconvexity) a weaker condition than concavity (convexity). In Fig. 12.4, we contrast strict concavity against strict quasiconcavity for the two-variable case. As drawn, both surfaces depict increasing functions, as they contain only the ascending portions of a dome and a bell, respectively. The surface in Fig. 12.4*a* is strictly concave, but the one in Fig. 12.4*b* is certainly not, since it contains convex portions near the base of the bell. Yet it is strictly quasiconcave; all the arcs on the surface, exemplified by MN and $M'N'$, satisfy the condition that all the points on each arc between the two end points are higher than the lower end point. Returning to Fig. 12.4*a*, we should note that the surface therein is also strictly quasiconcave. Although we have not drawn any illustrative arcs MN and $M'N'$ in Fig. 12.4*a*, it is not difficult to check that all possible arcs do indeed satisfy the condition for strict quasiconcavity. In general, a strictly concave function must be strictly quasiconcave, although the converse is not true. We shall demonstrate this more formally in the paragraphs that follow.

Algebraic Definition

The preceding geometric characterization can be translated into an algebraic definition for easier generalization to higher-dimensional cases:

A function f is $\left\{\begin{array}{l}\text{quasiconcave}\\ \text{quasiconvex}\end{array}\right.$ iff, for any pair of distinct points u and v in the (convex-set) domain of f, and for $0 < \theta < 1$,

$$f(v) \geq f(u) \Rightarrow f[\theta u + (1-\theta)v] \left\{\begin{array}{l}\geq f(u)\\ \leq f(v)\end{array}\right\} \qquad (12.20)$$

To adapt this definition to *strict* quasiconcavity and quasiconvexity, the two weak inequalities on the right should be changed into strict inequalities $\left\{ \begin{matrix} > f(u) \\ < f(v) \end{matrix} \right\}$. You may find it instructive to compare (12.20) with (11.20).

From this definition, the following three theorems readily follow. These will be stated in terms of a function $f(x)$, where x can be interpreted as a vector of variables, $x = (x_1, \ldots, x_n)$.

Theorem I (negative of a function) If $f(x)$ is quasiconcave (strictly quasiconcave), then $-f(x)$ is quasiconvex (strictly quasiconvex).

Theorem II (concavity versus quasiconcavity) Any concave (convex) function is quasiconcave (quasiconvex), but the converse is not true. Similarly, any strictly concave (strictly convex) function is strictly quasiconcave (strictly quasiconvex), but the converse is not true.

Theorem III (linear function) If $f(x)$ is a linear function, then it is quasiconcave as well as quasiconvex.

Theorem I follows from the fact that multiplying an inequality by -1 reverses the sense of inequality. Let $f(x)$ be quasiconcave, with $f(v) \geq f(u)$. Then, by (12.20), $f[\theta u + (1-\theta)v] \geq f(u)$. As far as the function $-f(x)$ is concerned, however, we have (after multiplying the two inequalities through by -1) $-f(u) \geq -f(v)$ and $-f[\theta u + (1-\theta)v] \leq -f(u)$. Interpreting $-f(u)$ as the height of point N, and $-f(v)$ as the height of M, we see that the function $-f(x)$ satisfies the condition for quasiconvexity in (12.20). This proves one of the four cases cited in Theorem I; the proofs for the other three are similar.

For Theorem II, we shall only prove that concavity implies quasiconcavity. Let $f(x)$ be concave. Then, by (11.20),

$$f[\theta u + (1-\theta)v] \geq \theta f(u) + (1-\theta) f(v)$$

Now assume that $f(v) \geq f(u)$; then any weighted average of $f(v)$ and $f(u)$ cannot possibly be less than $f(u)$, i.e.,

$$\theta f(u) + (1-\theta) f(v) \geq f(u)$$

Combining these two results, we find that, by transitivity,

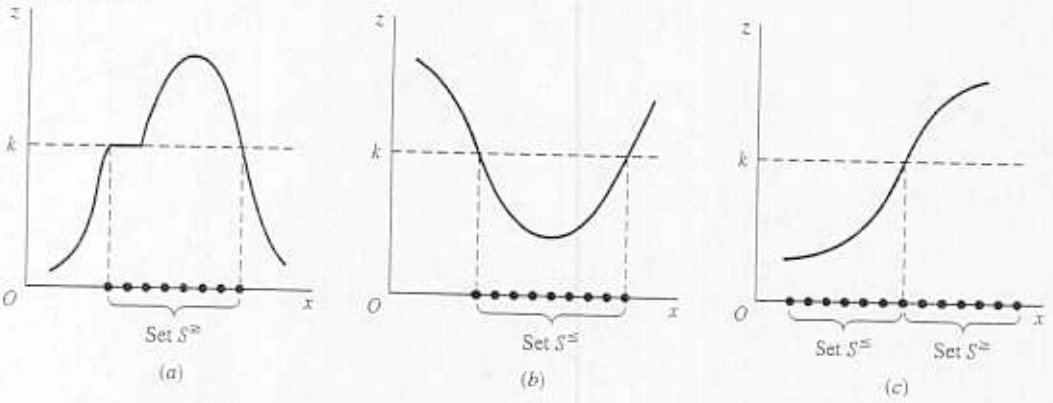
$$f[\theta u + (1-\theta)v] \geq f(u) \qquad \text{for } f(v) \geq f(u)$$

which satisfies the definition of quasiconcavity in (12.20). Note, however, that the condition for quasiconcavity cannot guarantee concavity.

Once Theorem II is established, Theorem III follows immediately. We already know that a linear function is both concave and convex, though not strictly so. In view of Theorem II, a linear function must also be both quasiconcave and quasiconvex, though not strictly so.

In the case of concave and convex functions, there is a useful theorem to the effect that the sum of concave (convex) functions is also concave (convex). Unfortunately, this theorem cannot be generalized to quasiconcave and quasiconvex functions. For instance, a sum of two quasiconcave functions is *not necessarily* quasiconcave (see Exercise 12.4-3).

FIGURE 12.5



Sometimes it may prove easier to check quasiconcavity and quasiconvexity by the following alternative definition:

A function $f(x)$, where x is a vector of variables, is $\left\{\begin{array}{l}\text{quasiconcave}\\ \text{quasiconvex}\end{array}\right\}$ iff, for any constant k, the set

$$\left\{\begin{array}{l} S^{\geq} \equiv \{x \mid f(x) \geq k\} \\ S^{\leq} \equiv \{x \mid f(x) \leq k\} \end{array}\right\} \text{ is a convex set} \qquad (12.21)$$

The sets $S^{\geq}$ and $S^{\leq}$, which are subsets of the domain, were introduced earlier (Fig. 11.10) to show that a convex function (or even a concave function) can give rise to a convex set. Here we are employing these two sets as tests for quasiconcavity and quasiconvexity. The three functions in Fig. 12.5 all contain concave as well as convex segments and hence are neither convex nor concave. But the function in Fig. 12.5*a* is quasiconcave, because for any value of k (only one of which has been illustrated), the set $S^{\geq}$ is convex. The function in Fig. 12.5*b* is, on the other hand, quasiconvex since the set $S^{\leq}$ is convex. The function in Fig. 12.5*c*—a monotonic function—differs from the other two in that both $S^{\geq}$ and $S^{\leq}$ are convex sets. Hence that function is quasiconcave as well as quasiconvex.

Note that while (12.21) can be used to check quasiconcavity and quasiconvexity, it is incapable of distinguishing between strict and nonstrict varieties of these properties. Note, also, that the defining properties in (12.21) are in themselves not sufficient for concavity and convexity, respectively. In particular, given a concave function which must perforce be quasiconcave, we can conclude that $S^{\geq}$ is a convex set; but given that $S^{\geq}$ is a convex set, we can conclude only that the function f is *quasi*concave (but not necessarily concave).

Example 1 Check $z = x^2$ $(x \geq 0)$ for quasiconcavity and quasiconvexity. This function is easily verified geometrically to be convex, in fact strictly so. Hence it is quasiconvex. Interestingly, it is also quasiconcave. For its graph—the right half of a U-shaped curve, initiating from the point of origin and increasing at an increasing rate—is, similarly to Fig. 12.5*c*, capable of generating a convex $S^{\geq}$ as well as a convex $S^{\leq}$.

If we wish to apply (12.20) instead, we first let u and v be any two distinct nonnegative values of x. Then

$$f(u) = u^2 \quad f(v) = v^2 \quad \text{and} \quad f[\theta u + (1-\theta)v] = [\theta u + (1-\theta)v]^2$$

Suppose that $f(v) \geq f(u)$, that is, $v^2 \geq u^2$; then $v \geq u$, or more specifically, $v > u$ (since u and v are distinct). Inasmuch as the weighted average $[\theta u + (1-\theta)v]$ must lie between u and v, we may write the continuous inequality

$$v^2 > [\theta u + (1-\theta)v]^2 > u^2 \qquad \text{for } 0 < \theta < 1$$

or

$$f(v) > f[\theta u + (1-\theta)v] > f(u) \qquad \text{for } 0 < \theta < 1$$

By (12.20), this result makes the function f *both* quasiconcave and quasiconvex—indeed strictly so.

Example 2

Show that $z = f(x, y) = xy$ (with $x, y \geq 0$) is quasiconcave. We shall use the criterion in (12.21) and establish that the set $S^{\geq} = \{(x, y) \mid xy \geq k\}$ is a convex set for any k. For this purpose, we set $xy = k$ to obtain an isovalue curve for each value of k. Like x and y, k should be nonnegative. In case $k > 0$, the isovalue curve is a rectangular hyperbola in the first quadrant of the xy plane. The set $S^{\geq}$, consisting of all the points on or above a rectangular hyperbola, is a convex set. In the other case, with $k = 0$, the isovalue curve as defined by $xy = 0$ is L-shaped, with the L coinciding with the nonnegative segments of the x and y axes. The set $S^{\geq}$, consisting this time of the entire nonnegative quadrant, is again a convex set. Thus, by (12.21), the function $z = xy$ (with $x, y \geq 0$) is quasiconcave.

You should be careful not to confuse the shape of the isovalue curves $xy = k$ (which is defined in the xy plane) with the shape of the surface $z = xy$ (which is defined in the xyz space). The characteristic of the z surface (quasiconcave in 3-space) is what we wish to ascertain; the shape of the isovalue curves (convex in 2-space for positive k) is of interest here only as a means to delineate the sets $S^{\geq}$ in order to apply the criterion in (12.21).

Example 3

Show that $z = f(x, y) = (x-a)^2 + (y-b)^2$ is quasiconvex. Let us again apply (12.21). Setting $(x-a)^2 + (y-b)^2 = k$, we see that k must be nonnegative. For each k, the isovalue curve is a circle in the xy plane with its center at (a, b) and with radius $\sqrt{k}$. Since $S^{\leq} = \{(x, y) \mid (x-a)^2 + (y-b)^2 \leq k\}$ is the set of all points on or inside a circle, it constitutes a convex set. This is true even when $k = 0$—when the circle degenerates into a single point, (a, b)—since by convention a single point is considered as a convex set. Thus the given function is quasiconvex.

Differentiable Functions

The definitions (12.20) and (12.21) do not require differentiability of the function f. If f is differentiable, however, quasiconcavity and quasiconvexity can alternatively be defined in terms of its first derivatives:

A differentiable function of one variable, $f(x)$, is $\left\{\begin{matrix}\text{quasiconcave}\\ \text{quasiconvex}\end{matrix}\right\}$ iff, for any pair of distinct points u and v in the domain,

$$f(v) \geq f(u) \Rightarrow \left\{\begin{matrix} f'(u)(v-u) \\ f'(v)(v-u)\end{matrix}\right\} \geq 0 \qquad (12.22)$$

Quasiconcavity and quasiconvexity will be *strict*, if the weak inequality on the right is changed to the strict inequality > 0. When there are two or more independent variables, the definition is to be modified as follows:

A differentiable function $f(x_1, \ldots, x_n)$ is $\left\{\begin{matrix}\text{quasiconcave}\\ \text{quasiconvex}\end{matrix}\right\}$ iff, for any two distinct points

$u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in the domain,

$$f(v) \geq f(u) \Rightarrow \begin{vmatrix} \sum_{j=1}^{n} f_j(u)(v_j - u_j) \\ \sum_{j=1}^{n} f_j(v)(v_j - u_j) \end{vmatrix} \geq 0 \qquad (12.22')$$

where $f_j \equiv \partial f / \partial x_j$, to be evaluated at u or v as the case may be.

Again, for *strict* quasiconcavity and quasiconvexity, the weak inequality on the right should be changed to the strict inequality > 0.

Finally, if a function $z = f(x_1, \ldots, x_n)$ is twice continuously differentiable, quasiconcavity and quasiconvexity can be checked by means of the first and second partial derivatives of the function, arranged into the bordered determinant

$$|B| = \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix} \qquad (12.23)$$

This bordered determinant resembles the bordered Hessian $|\bar{H}|$ introduced in Sec. 12.3. But unlike the latter, the border in $|B|$ is composed of the first derivatives of the function f rather than an extraneous constraint function g. It is because $|B|$ depends exclusively on the derivatives of function f itself that we can use $|B|$, along with its leading principal minors

$$|B_1| = \begin{vmatrix} 0 & f_1 \\ f_1 & f_{11} \end{vmatrix} \quad |B_2| = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} \qquad |B_n| = |B| \quad (12.24)$$

to characterize the configuration of that function.

We shall state here two conditions; one is necessary, and the other is sufficient. Both relate to quasiconcavity on a domain consisting only of the *nonnegative orthant* (the n-dimensional analog of the nonnegative quadrant), that is, with $x_1, \ldots, x_n \geq 0$.†

For $z = f(x_1, \ldots, x_n)$ to be quasiconcave on the nonnegative orthant, it is necessary that

$$|B_1| \leq 0, \quad |B_2| \geq 0, \qquad |B_n| \begin{Bmatrix} \leq \\ \geq \end{Bmatrix} 0 \text{ if } n \text{ is } \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix} \qquad (12.25)$$

wherever the partial derivatives are evaluated in the nonnegative orthant.

† Whereas concavity (convexity) of a function on a convex domain can always be extended to concavity (convexity) over the entire space, quasiconcavity and quasiconvexity cannot. For instance, our conclusions in Examples 1 and 2 will not hold if the variables are allowed to take negative values. The two conditions given here are based on Kenneth J. Arrow and Alain C. Enthoven, "Quasi-Concave Programming," *Econometrica*, October 1961, p. 797 (Theorem 5), and Akira Takayama, *Analytical Methods in Economics*, University of Michigan Press, 1993, p. 65 (Theorem 1.12).

A sufficient condition for f to be strictly quasiconcave on the nonnegative orthant is that

$$|B_1| < 0, \quad |B_2| > 0, \qquad |B_n| \begin{Bmatrix} < \\ > \end{Bmatrix} 0 \text{ if } n \text{ is } \begin{cases} \text{odd} \\ \text{even} \end{cases} \tag{12.26}$$

wherever the partial derivatives are evaluated in the nonnegative orthant.

Note that the condition $|B_1| \leq 0$ in (12.25) is automatically satisfied because $|B_1| = -f_1^2$; it is listed here only for the sake of symmetry. So is the condition $|B_1| < 0$ in (12.26).

Example 4 The function $z = f(x_1, x_2) = x_1 x_2$ $(x_1, x_2 \geq 0)$ is quasiconcave (cf. Example 2). We shall now check this by (12.22′). Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be any two points in the domain. Then $f(u) = u_1 u_2$ and $f(v) = v_1 v_2$. Assume that

$$f(v) \geq f(u) \quad \text{or} \quad v_1 v_2 \geq u_1 u_2 \quad (v_1, v_2, u_1, u_2 \geq 0) \tag{12.27}$$

Since the partial derivatives of f are $f_1 = x_2$ and $f_2 = x_1$, (12.22′) amounts to the condition that

$$f_1(u)(v_1 - u_1) + f_2(u)(v_2 - u_2) = u_2(v_1 - u_1) + u_1(v_2 - u_2) \geq 0$$

or, upon rearrangement,

$$u_2(v_1 - u_1) \geq u_1(u_2 - v_2) \tag{12.28}$$

We need to consider four possibilities regarding the values of u_1 and u_2. First, if $u_1 = u_2 = 0$, then (12.28) is trivially satisfied. Second, if $u_1 = 0$ but $u_2 > 0$, then (12.28) reduces to the condition $u_2 v_1 \geq 0$, which is again satisfied since u_2 and v_1 are both nonnegative. Third, if $u_1 > 0$ and $u_2 = 0$, then (12.28) reduces to the condition $0 \geq -u_1 v_2$, which is still satisfied. Fourth and last, suppose that u_1 and u_2 are both positive, so that v_1 and v_2 are also positive. Subtracting $v_2 u_1$ from both sides of (12.27), we obtain

$$v_2(v_1 - u_1) \geq u_1(u_2 - v_2) \tag{12.29}$$

Three subpossibilities now present themselves:

1. If $u_2 = v_2$, then $v_1 \geq u_1$. In fact, we should have $v_1 > u_1$ since (u_1, u_2) and (v_1, v_2) are distinct points. The fact that $u_2 = v_2$ and $v_1 > u_1$ implies that condition (12.28) is satisfied.
2. If $u_2 > v_2$, then we must also have $v_1 > u_1$ by (12.29). Multiplying both sides of (12.29) by u_2/v_2, we get

$$u_2(v_1 - u_1) \geq \frac{u_2}{v_2} u_1(u_2 - v_2) > u_1(u_2 - v_2) \quad \left[\text{since} \frac{u_2}{v_2} > 1\right] \tag{12.30}$$

   Thus (12.28) is again satisfied.
3. The final subpossibility is that $u_2 < v_2$, implying that u_2/v_2 is a positive fraction. In this case, the first line of (12.30) still holds. The second line also holds, but now for a different reason: a fraction (u_2/v_2) of a negative number $(u_2 - v_2)$ is greater than the latter number itself.

Inasmuch (12.28) is satisfied in every possible situation that can arise, the function $z = x_1 x_2$ ($x_1, x_2 \geq 0$) is quasiconcave. Therefore, the necessary condition (12.25) should hold. Because the partial derivatives of f are

$$f_1 = x_2 \quad f_2 = x_1 \quad f_{11} = f_{22} = 0 \quad f_{12} = f_{21} = 1$$

the relevant leading principal minors turn out to be

$$|B_1| = \begin{vmatrix} 0 & x_2 \\ x_2 & 0 \end{vmatrix} = -x_2^2 \leq 0 \qquad |B_2| = \begin{vmatrix} 0 & x_2 & x_1 \\ x_2 & 0 & 1 \\ x_1 & 1 & 0 \end{vmatrix} = 2x_1 x_2 \geq 0$$

Thus (12.25) is indeed satisfied. Note, however, that the sufficient condition (12.26) is satisfied only over the positive orthant.

Example 5 Show that $z = f(x, y) = x^a y^b$ ($x, y > 0$; $0 < a, b < 1$) is strictly quasiconcave. The partial derivatives of this function are

$$f_x = ax^{a-1}y^b \quad f_y = bx^a y^{b-1}$$
$$f_{xx} = a(a-1)x^{a-2}y^b \quad f_{xy} = f_{yx} = abx^{a-1}y^{b-1} \quad f_{yy} = b(b-1)x^a y^{b-2}$$

Thus the leading principal minors of $|B|$ have the following signs:

$$|B_1| = \begin{vmatrix} 0 & f_x \\ f_x & f_{xx} \end{vmatrix} = -(ax^{a-1}y^b)^2 < 0$$

$$|B_2| = \begin{vmatrix} 0 & f_x & f_y \\ f_x & f_{xx} & f_{xy} \\ f_y & f_{yx} & f_{yy} \end{vmatrix} = [2a^2b^2 - a(a-1)b^2 - a^2b(b-1)]x^{3a-2}y^{3b-2} > 0$$

This satisfies the sufficient condition for strict quasiconcavity in (12.26).

A Further Look at the Bordered Hessian

The bordered determinant $|B|$, as defined in (12.23), differs from the bordered Hessian

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & Z_{11} & Z_{12} & \cdots & Z_{1n} \\ g_2 & Z_{21} & Z_{22} & \cdots & Z_{2n} \\ \cdots & \cdots & \cdots & & \\ g_n & Z_{n1} & Z_{n2} & \cdots & Z_{nn} \end{vmatrix}$$

in two ways: (1) the border elements in $|B|$ are the first-order partial derivatives of function f rather than g; and (2) the remaining elements in $|B|$ are the second-order partial derivatives of f rather than the Lagrangian function Z. However, in the special case of a linear constraint equation, $g(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n = c$—a case frequently encountered in economics (see Sec. 12.5)—Z_{ij} reduces to f_{ij}. For then the Lagrangian function is

$$Z = f(x_1, \ldots, x_n) + \lambda(c - a_1x_1 - \cdots - a_nx_n)$$

so that

$$Z_j = f_j - \lambda a_j \qquad \text{and} \qquad Z_{ij} = f_{ij}$$

FIGU

Turning to the borders, we note that the linear constraint function yields the first derivative $g_j = a_j$. Moreover, when the first-order condition is satisfied, we have $Z_j = f_j - \lambda a_j = 0$, so that $f_j = \lambda a_j$, or $f_j = \lambda g_j$. Thus the border in $|B|$ is simply that of $|\bar{H}|$ multiplied by a positive scalar λ. By factoring out λ successively from the horizontal and vertical borders of $|\bar{H}|$ (see Sec. 5.3, Example 5), we have

$$|B| = \lambda^2 |\bar{H}|$$

Consequently, in the linear-constraint case, the two bordered determinants always possess the same sign at the stationary point of Z. By the same token, the leading principal minors $|B_i|$ and $|\bar{H}_i|$ $(i = 1, \ldots, n)$ must also share the same sign at that point. It then follows that if the bordered determinant $|B|$ satisfies the sufficient condition for strict quasiconcavity in (12.26), the bordered Hessian $|\bar{H}|$ must then satisfy the second-order sufficient condition for constrained maximization in Table 12.1.

Absolute versus Relative Extrema

A more comprehensive picture of the relationship between quasiconcavity and second-order conditions is presented in Fig. 12.6. (A suitable modification will adapt the figure for quasiconvexity.) Constructed in the same spirit—and to be read in the same manner—as Fig. 11.5, this figure relates quasiconcavity to *absolute* as well as *relative* constrained maxima of a twice-differentiable function $z = f(x_1, \ldots, x_n)$. The three ovals in the upper part summarize the first- and second-order conditions for a relative constrained maximum. And the rectangles in the middle column, like those in Fig. 11.5, tie the concepts of relative maximum, absolute maximum, and unique absolute maximum to one another.

But the really interesting information are those in the two diamonds and the elongated $\Rightarrow$ symbols passing through them. The one on the left tells us that, once the first-order condition is satisfied, and if the two provisos listed in the diamond are also satisfied, we have a sufficient condition for an absolute constrained maximum. The first proviso is that the function f be explicitly quasiconcave—a new term which we must hasten to define.

A quasiconcave function f is *explicitly quasiconcave* if it has the further property that

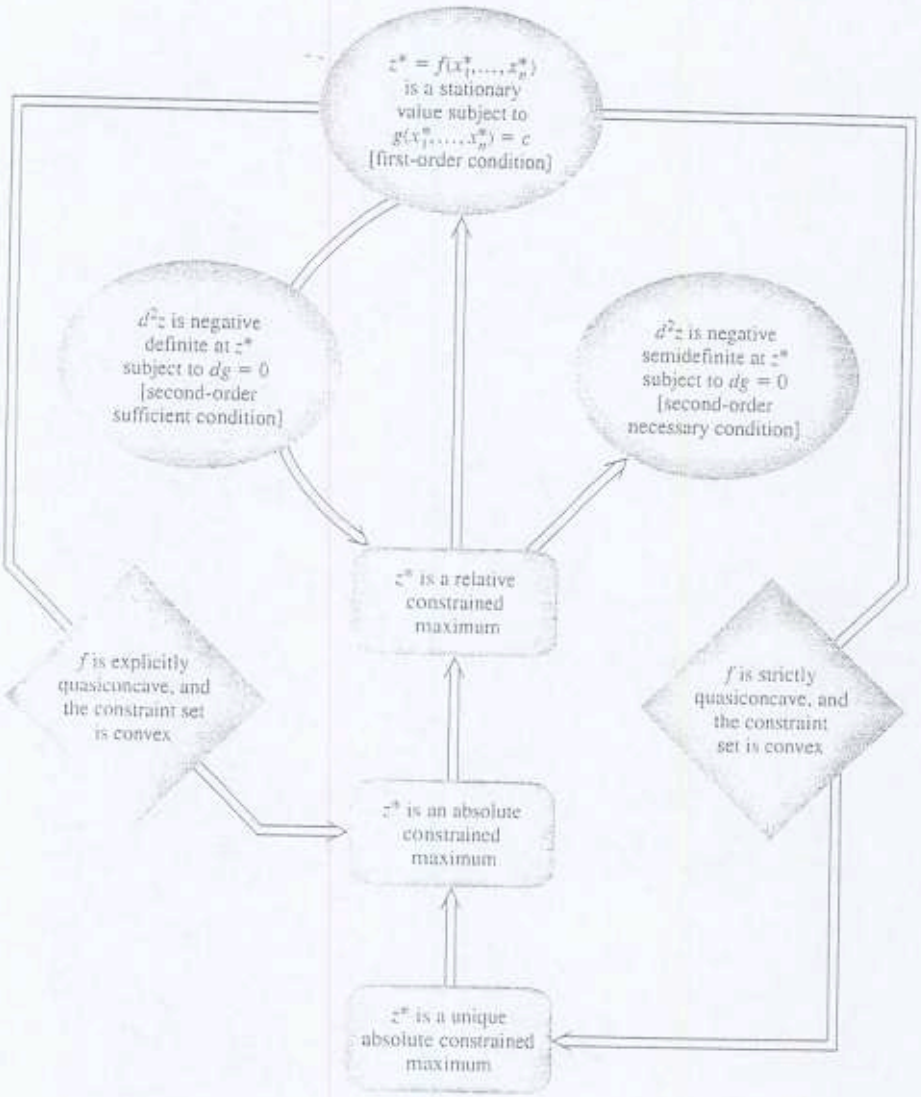
$$f(v) > f(u) \Rightarrow f[\theta u + (1 - \theta)v] > f(u)$$

This defining property means that whenever a point on the surface, $f(v)$, is higher than another, $f(u)$, then all the intermediate points—the points on the surface lying directly above line segment uv in the domain—must also be higher than $f(u)$. What such a stipulation does is to rule out any *horizontal* plane segments on the surface except for a plateau at the top of the surface.† Note that the condition for *explicit* quasiconcavity is not as strong as the condition for *strict* quasiconcavity, since the latter requires $f[\theta u + (1 - \theta)v] > f(u)$ even for $f(v) = f(u)$, implying that *non*horizontal plane segments are ruled out, too.‡ The other

† Let the surface contain a horizontal plane segment P such that $f(u) \in P$ and $f(v) \notin P$. Then those intermediate points that are located on P will be of equal height to $f(u)$, thereby violating the first proviso.

‡ Let the surface contain a slanted plane segment P' such that $f(u) = f(v)$ are both located on P'. Then all the intermediate points will also be on P' and be of equal height to $f(u)$, thereby violating the cited requirement for strict quasiconcavity.

FIGURE 12.6



proviso in the left-side diamond is that the set $[(x_1, \ldots, x_n) \mid g(x_1, \ldots, x_n) = c]$ be convex. When both provisos are met, we shall be dealing with that portion of a bell-shaped, horizontal-segment-free surface (or hypersurface) lying directly above a convex set in the domain. A local maximum found on such a subset of the surface must be an absolute constrained maximum.

The diamond on the right in Fig. 12.6 involves the stronger condition of *strict* quasiconcavity. A strictly quasiconcave function must be explicitly quasiconcave, although the converse is not true. Hence, when strict quasiconcavity replaces explicit quasiconcavity, an

absolute constrained maximum is still ensured. But this time that absolute constrained maximum must also be unique, since the absence of any plane segment anywhere on the surface decidedly precludes the possibility of multiple constrained maxima.

EXERCISE 12.4

1. Draw a strictly quasiconcave curve $z = f(x)$ which is
   (*a*) also quasiconvex (*d*) not concave
   (*b*) not quasiconvex (*e*) neither concave nor convex
   (*c*) not convex (*f*) both concave and convex
2. Are the following functions quasiconcave? Strictly so? First check graphically, and then algebraically by (12.20). Assume that $x \geq 0$.
   (*a*) $f(x) = a$ (*b*) $f(x) = a + bx$ $(b > 0)$ (*c*) $f(x) = a + cx^2$ $(c < 0)$
3. (*a*) Let $z = f(x)$ plot as a negatively sloped curve shaped like the right half of a bell in the first quadrant, passing through the points (0, 5), (2, 4), (3, 2), and (5, 1). Let $z = g(x)$ plot as a positively sloped 45° line. Are $f(x)$ and $g(x)$ quasiconcave?
   (*b*) Now plot the sum $f(x) + g(x)$. Is the sum function quasiconcave?
4. By examining their graphs, and using (12.21), check whether the following functions are quasiconcave, quasiconvex, both, or neither:
   (*a*) $f(x) = x^3 - 2x$ (*b*) $f(x_1, x_2) = 6x_1 - 9x_2$ (*c*) $f(x_1, x_2) = x_2 - \ln x_1$
5. (*a*) Verify that a cubic function $z = ax^3 + bx^2 + cx + d$ is in general neither quasiconcave nor quasiconvex.
   (*b*) Is it possible to impose restrictions on the parameters such that the function becomes both quasiconcave and quasiconvex for $x \geq 0$?
6. Use (12.22) to check $z = x^2 (x \geq 0)$ for quasiconcavity and quasiconvexity.
7. Show that $z = xy$ $(x, y \geq 0)$ is not quasiconvex.
8. Use bordered determinants to check the following functions for quasiconcavity and quasiconvexity:
   (*a*) $z = -x^2 - y^2$ $(x, y > 0)$ (*b*) $z = -(x + 1)^2 - (y + 2)^2$ $(x, y > 0)$

12.5 Utility Maximization and Consumer Demand

The maximization of a utility function was cited in Sec. 12.1 as an example of constrained optimization. Let us now reexamine this problem in more detail. For simplicity, we shall still allow our hypothetical consumer the choice of only two goods, both of which have continuous, positive marginal-utility functions. The prices of both goods are market-determined, hence exogenous, although in this section we shall omit the zero subscript from the price symbols. If the purchasing power of the consumer is a given amount B (for budget), the problem posed will be that of maximizing a smooth utility (index) function

$$U = U(x, y) \qquad (U_x, U_y > 0)$$

subject to

$$xP_x + yP_y = B$$

The consumer also faces an environmental constraint on her total output of both goods. The environmental constraint is given by $x + y \leq 20$.

(*a*) Write out the Kuhn-Tucker first-order conditions.

(*b*) Find the consumer's optimal x and y. Identify which constraints are binding.

3. An electric company is setting up a power plant in a foreign country, and it has to plan its capacity. The peak-period demand for power is given by $P_1 = 400 - Q_1$ and the off-peak demand is given by $P_2 = 380 - Q_2$. The variable cost is 20 per unit (paid in both markets) and capacity costs 10 per unit which is only paid once and is used in both periods.

(*a*) Write out the Lagrangian and Kuhn-Tucker conditions for this problem.

(*b*) Find the optimal outputs and capacity for this problem.

(*c*) How much of the capacity is paid for by each market (i.e., what are the values of λ_1 and λ_2)?

(*d*) Now suppose capacity cost is 30 cents per unit (paid only once). Find quantities, capacity, and how much of the capacity is paid for by each market (i.e., λ_1 and λ_2).

13.4 Sufficiency Theorems in Nonlinear Programming

In the previous sections, we have introduced the Kuhn-Tucker conditions and illustrated their applications *as necessary* conditions in optimization problems with inequality constraints. Under certain circumstances, the Kuhn-Tucker conditions can also be taken as sufficient conditions.

The Kuhn-Tucker Sufficiency Theorem: Concave Programming

In classical optimization problems, the sufficient conditions for maximum and minimum are traditionally expressed in terms of the signs of second-order derivatives or differentials. As we have shown in Sec. 11.5, however, these second-order conditions are closely related to the concepts of concavity and convexity of the objective function. Here, in nonlinear programming, the sufficient conditions can also be stated directly in terms of concavity and convexity. And, in fact, these concepts will be applied not only to the objective function $f(x)$ but to the constraint functions $g^i(x)$ as well.

For the *maximization* problem, Kuhn and Tucker offer the following statement of sufficient conditions (sufficiency theorem):

Given the nonlinear programming problem

$$\begin{array}{lll} \text{Maximize} & \pi = f(x) & \\ \text{subject to} & g^i(x) \leq r_i & (i = 1, 2, \ldots, m) \\ \text{and} & x \geq 0 & \end{array}$$

if the following conditions are satisfied:

(*a*) the objective function $f(x)$ is differentiable and *concave* in the nonnegative orthant

(*b*) each constraint function $g^i(x)$ is differentiable and *convex* in the nonnegative orthant

(*c*) the point x^* satisfies the Kuhn-Tucker maximum conditions

then x^* gives a global maximum of $\pi = f(x)$.

Note that, in this theorem, the constraint qualification is nowhere mentioned. This is because we have already assumed, in condition (*c*), that the Kuhn-Tucker conditions are

satisfied at x^* and, consequently, the question of the constraint qualification is no longer an issue.

As it stands, the above theorem indicates that conditions (a), (b), and (c) are sufficient to establish x^* to be an optimal solution. Looking at it differently, however, we may also interpret it to mean that given (a) and (b), then the Kuhn-Tucker maximum conditions are sufficient for a maximum. In the preceding section, we learned that the Kuhn-Tucker conditions, though not necessary *per se,* become necessary when the constraint qualification is satisfied. Combining this information with the sufficiency theorem, we may now state that if the constraint qualification is satisfied and if conditions (a) and (b) are realized, then the Kuhn-Tucker maximum conditions will be *necessary-and-sufficient* for a maximum. This would be the case, for instance, when all the constraints are linear inequalities, which is sufficient for satisfying the constraint qualification.

The maximization problem dealt with in the sufficiency theorem above is often referred to as *concave programming*. This name arises because Kuhn and Tucker adopt the $\geq$ inequality instead of the $\leq$ inequality in every constraint, so that condition (b) would require the $g^i(x)$ functions to be *all concave,* like the $f(x)$ function. But we have modified the formulation in order to convey the idea that in a maximization problem, a constraint is imposed to "rein in" (hence, $\leq$) the attempt to ascend to higher points on the objective function. Though different in form, the two formulations are equivalent in substance. For brevity, we omit the proof.

As stated above, the sufficiency theorem deals only with maximization problems. But adaptation to *minimization* problems is by no means difficult. Aside from the appropriate changes in the theorem to reflect the reversal of the problem itself, all we have to do is to interchange the two words *concave* and *convex* in conditions (a) and (b) and to use the Kuhn-Tucker *minimum* conditions in condition (c). (See Exercise 13.4-1.)

The Arrow-Enthoven Sufficiency Theorem: Quasiconcave Programming

To apply the Kuhn-Tucker sufficiency theorem, certain concavity-convexity specifications must be met. These constitute quite stringent requirements. In another sufficiency theorem—the Arrow-Enthoven sufficiency theorem[†]—these specifications are relaxed to the extent of requiring only *quasiconcavity* and *quasiconvexity* in the objective and constraint functions. With the requirements thus weakened, the scope of applicability of the sufficient conditions is correspondingly widened.

In the original formulation of the Arrow-Enthoven paper, with a maximization problem and with constraints in the $\geq$ form, the $f(x)$ and $g^i(x)$ functions must uniformly be quasiconcave in order for their theorem to be applicable. This gives rise to the name *quasiconcave programming*. In our discussion here, however, we shall again use the $\leq$ inequality in the constraints of a maximization problem and the $\geq$ inequality in the minimization problem.

The theorem is as follows:

Given the nonlinear programming problem

$$\begin{array}{lll} \text{Maximize} & \pi = f(x) & \\ \text{subject to} & g^i(x) \leq r_i & (i = 1, 2, \ldots, m) \\ \text{and} & x \geq 0 & \end{array}$$

[†] Kenneth J. Arrow and Alain C. Enthoven, "Quasi-concave Programming," *Econometrica,* October, 1961, pp. 779–800.

if the following conditions are satisfied:

(a) the objective function $f(x)$ is differentiable and *quasiconcave* in the nonnegative orthant

(b) each constraint function $g^i(x)$ is differentiable and *quasiconvex* in the nonnegative orthant

(c) the point x^* satisfies the Kuhn-Tucker maximum conditions

(d) any *one* of the following is satisfied:

(d-i) $f_j(x^*) < 0$ for at least one variable x_j

(d-ii) $f_j(x^*) > 0$ for some variable x_j that can take on a positive value without violating the constraints

(d-iii) the n derivatives $f_j(x^*)$ are not all zero, and the function $f(x)$ is twice differentiable in the neighborhood of x^* [i.e., all the second-order partial derivatives of $f(x)$ exist at x^*]

(d-iv) the function $f(x)$ is concave

then x^* gives a global maximum of $\pi = f(x)$.

Since the proof of this theorem is quite lengthy, we shall omit it here. However, we do want to call your attention to a few important features of this theorem. For one thing, while Arrow and Enthoven have succeeded in weakening the concavity-convexity specifications to their quasiconcavity-quasiconvexity counterparts, they find it necessary to append a new requirement, (d). Note, though, that only *one* of the four alternatives listed under (d) is required to form a complete set of sufficient conditions. In effect, therefore, the above theorem contains as many as *four* different sets of sufficient conditions for a maximum. In the case of (d-iv), with $f(x)$ concave, it would apear that the Arrow-Enthoven sufficiency theorem becomes identical with the Kuhn-Tucker sufficiency theorem. But this is not true. Inasmuch as Arrow and Enthoven only require the constraint functions $g^i(x)$ to be *quasiconvex,* their sufficient conditions are still weaker.

As stated, the theorem lumps together the conditions (a) through (d) as a set of sufficient conditions. But it is also possible to interpret it to mean that, when (a), (b), and (d) are satisfied, then the Kuhn-Tucker maximum conditions become sufficient conditions for a maximum. Furthermore, if the constraint qualification is also satisfied, then the Kuhn-Tucker conditions will become necessary-and-sufficient for a maximum.

Like the Kuhn-Tucker theorem, the Arrow-Enthoven theorem can be adapted with ease to the *minimization* framework. Aside from the obvious changes that are needed to reverse the direction of optimization, we simply have to interchange the words *quasiconcave* and *quasiconvex* in conditions (a) and (b), replace the Kuhn-Tucker maximum conditions by the minimum conditions, reverse the inequalities in (d-i) and (d-ii), and change the word *concave* to *convex* in (d-iv).

A Constraint-Qualification Test

It was mentioned in Sec. 13.2 that if all constraint functions are linear, then the constraint qualification is satisfied. In case the $g^i(x)$ functions are nonlinear, the following test offered by Arrow and Enthoven may prove useful in determining whether the constraint qualification is satisfied:

EXER

For a maximization problem, if

(*a*) every constraint function $g^j(x)$ is differentiable and quasiconvex

(*b*) there exists a point x^0 in the nonnegative orthant such that all the constraints are satisfied as strict inequalities at x^0

(*c*) *one* of the following is true:

(*c-i*) every $g^j(x)$ function is convex

(*c-ii*) the partial derivatives of every $g^j(x)$ are not all zero when evaluated at every point x in the feasible region

then the constraint qualification is satisfied.

Again, this test can be adapted to the minimization problem with ease. To do so, just change the word *quasiconvex* to *quasiconcave* in condition (*a*), and change the word *convex* to *concave* in (*c-i*).

EXERCISE 13.4

1. Given: Minimize $C = F(x)$
   subject to $G^j(x) \geq r_j \quad (j = 1, 2, \ldots, m)$
   and $x \geq 0$
   (*a*) Convert it into a maximization problem.
   (*b*) What in the present problem are the equivalents of the f and g^j functions in the Kuhn-Tucker sufficiency theorem?
   (*c*) Hence, what concavity-convexity conditions should be placed on the F and G^j functions to make the sufficient conditions for a maximum applicable here?
   (*d*) On the basis of the above, how would you state the Kuhn-Tucker sufficient conditions for a *minimum*?
2. Is the Kuhn-Tucker sufficiency theorem applicable to:
   (*a*) Maximize $\pi = x_1$
   subject to $x_1^2 + x_3^2 \leq 1$
   and $x_1, x_2 \geq 0$
   (*b*) Minimize $C = (x_1 - 3)^2 + (x_2 - 4)^2$
   subject to $x_1 + x_2 \geq 4$
   and $x_1, x_2 \geq 0$
   (*c*) Minimize $C = 2x_1 + x_2$
   subject to $x_1^2 - 4x_1 + x_2 \geq 0$
   and $x_1, x_2 \geq 0$
3. Which of the following functions are mathematically acceptable as the objective function of a *maximization* problem which qualifies for the application of the Arrow-Enthoven sufficiency theorem?
   (*a*) $f(x) = x^3 - 2x$
   (*b*) $f(x_1, x_2) = 6x_1 - 9x_2$
   (*c*) $f(x_1, x_2) = x_2 - \ln x_1$ (*Note:* See Exercise 12.4-4.)

4. Is the Arrow-Enthoven constraint qualification satisfied, given that the constraints of a *maximization* problem are:
   (*a*) $x_1^2 + (x_2 - 5)^2 \leq 4$ and $5x_1 + x_2 < 10$
   (*b*) $x_1 + x_2 \leq 8$ and $-x_1 x_2 \leq -8$ (*Note:* $-x_1 x_2$ is not convex.)

13.5 Maximum-Value Functions and the Envelope Theorem†

A maximum-value function is an objective function where the choice variables have been assigned their optimal values. These optimal values of the choice variables are, in turn, functions of the exogenous variables and parameters of the problem. Once the optimal values of the choice variables have been substituted into the original objective function, the function indirectly becomes a function of the parameters only (through the parameters' influence on the optimal values of the choice variables). Thus the maximum-value function is also referred to as the *indirect objective function*.

The Envelope Theorem for Unconstrained Optimization

What is the significance of the indirect objective function? Consider that in any optimization problem the direct objective function is maximized (or minimized) for a given set of parameters. The indirect objective function traces out all the maximum values of the objective function as these parameters vary. Hence the indirect objective function is an "envelope" of the set of optimized objective functions generated by varying the parameters of the model. For most students of economics the first illustration of this notion of an envelope arises in the comparison of short-run and long-run cost curves. Students are typically taught that the long-run average cost curve is an envelope of all the short-run average cost curves (what parameter is varying along the envelope in this case?). A formal derivation of this concept is one of the exercises we will be doing in this section.

To illustrate, consider the following unconstrained maximization problem with two choice variables x and y and one parameter ϕ:

$$\text{Maximize} \qquad U = f(x, y, \phi) \tag{13.27}$$

The first-order necessary condition is

$$f_x(x, y, \phi) = f_y(x, y, \phi) = 0 \tag{13.28}$$

If second-order conditions are met, these two equations implicitly define the solutions

$$x^* = x^*(\phi) \qquad y^* = y^*(\phi) \tag{13.29}$$

If we substitute these solutions into the objective function, we obtain a new function

$$V(\phi) = f(x^*(\phi),\ y^*(\phi), \phi) \tag{13.30}$$

where this function is the value of f when the values of x and y are those that maximize $f(x, y, \phi)$. Therefore, $V(\phi)$ is *the maximum-value function* (or indirect objective function).

† This section of the chapter presents an overview of the envelope theorem. A richer treatment of this topic can be found in Chap. 7 of *The Structure of Economics: A Mathematical Analysis* (3rd ed.) by Eugene Silberberg and Wing Suen (McGraw-Hill, 2001) on which parts of this section are based.