(*b*) Can we do the same for the following set of conditions?

$$x_j \frac{\partial Z}{\partial x_j} = 0 \qquad (j = 1, \ldots, n)$$

3. Based on the reasoning used in Prob. 2, which set (or sets) of conditions in (13.17) can be condensed into a single equation?
4. Suppose the problem is

   Minimize $\quad C = f(x_1, x_2, \ldots, x_n)$

   subject to $\quad g^i(x_1, x_2, \ldots, x_n) \geq r_i$

   and $\quad x_j \geq 0 \quad \begin{pmatrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{pmatrix}$

   Write the Lagrangian function, take the derivatives $\partial Z/\partial x_j$ and $\partial Z/\partial \lambda_i$ and write out the expanded version of the Kuhn-Tucker minimum conditions (13.17).
5. Convert the minimization problem in Prob. 4 into a maximization problem, formulate the Lagrangian function, take the derivatives with respect to x_j and λ_i, and apply the Kuhn-Tucker maximum conditions (13.16). Are the results consistent with those obtained in Prob. 4?

13.2 The Constraint Qualification

The Kuhn-Tucker conditions are necessary conditions *only if* a particular proviso is satisfied. That proviso, called the *constraint qualification*, imposes a certain restriction on the constraint functions of a nonlinear programming problem, for the specific purpose of ruling out certain irregularities on the boundary of the feasible set, that would invalidate the Kuhn-Tucker conditions should the optimal solution occur there.

Irregularities at Boundary Points

Let us first illustrate the nature of such irregularities by means of some concrete examples.

Example 1

Maximize $\quad \pi = x_1$

subject to $\quad x_2 - (1 - x_1)^3 \leq 0$

and $\quad x_1, x_2 \geq 0$

As shown in Fig. 13.2, the feasible region is the set of points that lie in the first quadrant on or below the curve $x_2 = (1 - x_1)^3$. Since the objective function directs us to maximize x_1, the optimal solution is the point (1, 0). But the solution fails to satisfy the Kuhn-Tucker maximum conditions. To check this, we first write the Lagrangian function

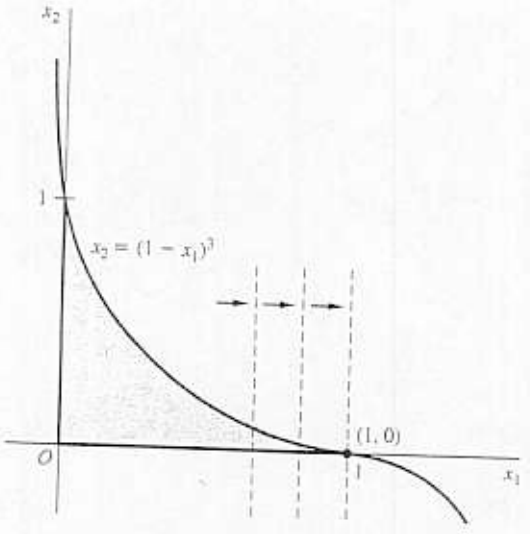
$$Z = x_1 + \lambda_1[-x_2 + (1 - x_1)^3]$$

As the first marginal condition, we should then have

$$\frac{\partial Z}{\partial x_1} = 1 - 3\lambda_1(1 - x_1)^2 \leq 0$$

In fact, since $x_1^* = 1$ is positive, complementary slackness requires that this derivative vanish when evaluated at the point (1, 0). However, the actual value we get happens to be $\partial Z/\partial x_1^* = 1$, thus violating the given marginal condition.

FIGURE 13.2



The reason for this anomaly is that the optimal solution (1, 0) occurs in this example at an outward-pointing *cusp*, which constitutes one type of irregularity that can invalidate the Kuhn-Tucker conditions at a boundary optimal solution. A *cusp* is a sharp point formed when a curve takes a sudden reversal in direction, such that the slope of the curve on one side of the point is the same as the slope of the curve on the other side of the point. Here, the boundary of the feasible region at first follows the constraint curve, but when the point (1, 0) is reached, it takes an abrupt turn westward and follows the trail of the horizontal axis thereafter. Since the slopes of both the curved side and the horizontal side of the boundary are zero at the point (1, 0), that point is a cusp.

Cusps are the most frequently cited culprits for the failure of the Kuhn-Tucker conditions, but the truth is that the presence of a cusp is neither necessary nor sufficient to cause those conditions to fail at an optimal solution. Examples 2 and 3 will confirm this.

Example 2 To the problem of Example 1, let us add a new constraint

$$2x_1 + x_2 \leq 2$$

whose border, $x_2 = 2 - 2x_1$, plots as a straight line with slope -2 which passes through the optimal point in Fig. 13.2. Clearly, the feasible region remains the same as before, and so does the optimal solution at the cusp. But if we write the new Lagrangian function

$$Z = x_1 + \lambda_1[-x_2 + (1 - x_1)^3] + \lambda_2[2 - 2x_1 - x_2]$$

and the marginal conditions

$$\frac{\partial Z}{\partial x_1} = 1 - 3\lambda_1(1 - x_1)^2 - 2\lambda_2 \leq 0$$

$$\frac{\partial Z}{\partial x_2} = -\lambda_1 - \lambda_2 \leq 0$$

$$\frac{\partial Z}{\partial \lambda_1} = -x_2 + (1 - x_1)^3 \geq 0$$

$$\frac{\partial Z}{\partial \lambda_2} = 2 - 2x_1 - x_2 \geq 0$$

it turns out that the values $x_1^* = 1$, $x_2^* = 0$, $\lambda_1^* = 1$, and $\lambda_2^* = \frac{1}{2}$ do satisfy these four inequalities, as well as the nonnegativity and complementary-slackness conditions. As a matter of fact, λ_1^* can be assigned any nonnegative value (not just 1), and all the conditions can still be satisfied—which goes to show that the optimal value of a Lagrange multiplier is not necessarily unique. More importantly, however, this example shows that the Kuhn-Tucker conditions can remain valid despite the cusp.

Example 3 The feasible region of the problem

$$\begin{array}{ll} \text{Maximize} & \pi = x_2 - x_1^2 \\ \text{subject to} & -\left(10 - x_1^2 - x_2\right)^3 \leq 0 \\ & -x_1 \leq -2 \\ \text{and} & x_1, x_2 \geq 0 \end{array}$$

as shown in Fig. 13.3, contains no cusp anywhere. Yet, at the optimal solution, (2, 6), the Kuhn-Tucker conditions nonetheless fail to hold. For, with the Lagrangian function

$$Z = x_2 - x_1^2 + \lambda_1\left(10 - x_1^2 - x_2\right)^3 + \lambda_2(-2 + x_1)$$

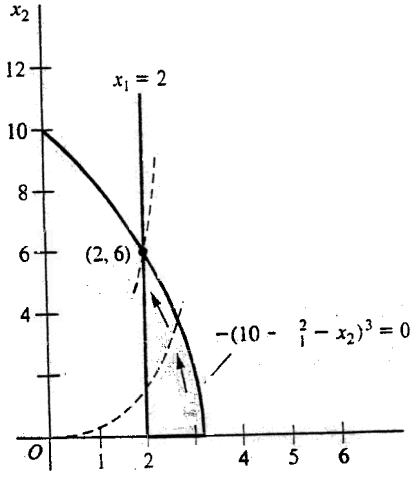
the second marginal condition would require that

$$\frac{\partial Z}{\partial x_2} = 1 - 3\lambda_1\left(10 - x_1^2 - x_2\right)^2 \leq 0$$

Indeed, since x_2^* is positive, this derivative should vanish when evaluated at the point (2, 6). But actually we get $\partial Z/\partial x_2 = 1$, regardless of the value assigned to λ_1. Thus the Kuhn-Tucker conditions can fail even in the absence of a cusp—nay, even when the feasible region is a convex set as in Fig. 13.3. The fundamental reason why cusps are neither necessary nor sufficient for the failure of the Kuhn-Tucker conditions is that the preceding irregularities referred to before relate, not to the shape of the feasible region per se, but to the forms of the constraint functions themselves.

Example

FIGURE 13.3



The Constraint Qualification

Boundary irregularities—cusp or no cusp—will not occur if a certain constraint qualification is satisfied.

To explain this, let $x^* \equiv (x_1^*, x_2^*, \ldots, x_n^*)$ be a boundary point of the feasible region and a possible candidate for a solution, and let $dx \equiv (dx_1, dx_2, \ldots, dx_n)$ represent a particular direction of movement from the said boundary point. The direction-of-movement interpretation of the vector dx is perfectly in line with our earlier interpretation of a vector as a directed line segment (an arrow), but here, the point of departure is the point x^* instead of the point of origin, and so the vector dx is *not* in the nature of a radius vector. We shall now impose two requirements on the vector dx. First, if the jth choice variable has a zero value at the point x^*, then we shall only permit a nonnegative change on the x_j axis, that is,

$$dx_j \geq 0 \qquad \text{if} \qquad x_j^* = 0 \tag{13.22}$$

Second, if the ith constraint is exactly satisfied at the point x^*, then we shall only allow values of $dx_1, \ldots, dx_n$ such that the value of the constraint function $g^i(x^*)$ will not increase (for a maximization problem) or will not decrease (for a minimization problem), that is,

$$dg^i(x^*) = g_1^i\,dx_1 + g_2^i\,dx_2 + \quad + g_n^i\,dx_n \quad \begin{cases} \leq 0 \text{ (max.)} \\ \geq 0 \text{ (min.)} \end{cases} \qquad \text{if} \qquad g^i(x^*) = r_i \tag{13.23}$$

where all the partial derivatives of g_j^i are to be evaluated at x^*. If a vector dx satisfies (13.22) and (13.23), we shall refer to it as a *test vector*. Finally, if there exists a differentiable arc that (1) emanates from the point x^*, (2) is contained entirely in the feasible region, and (3) is tangent to a given test vector, we shall call it a *qualifying arc* for that test vector. With this background, the constraint qualification can be stated simply as follows:

> The constraint qualification is satisfied if, for any point x^* on the boundary of the feasible region, there exists a qualifying arc for every test vector dx.

Example 4 We shall show that the optimal point (1, 0) of Example 1 in Fig. 13.2, which fails the Kuhn-Tucker conditions, also fails the constraint qualification. At that point, $x_2^* = 0$; thus the test vector must satisfy

$$dx_2 \geq 0 \qquad \text{[by (13.22)]}$$

Moreover, since the (only) constraint, $g^1 = x_2 - (1 - x_1)^3 \leq 0$, is exactly satisfied at (1, 0), we must let [by (13.23)]

$$g_1^1\,dx_1 + g_2^1\,dx_2 = 3(1 - x_1^*)^2\,dx_1 + dx_2 = dx_2 \leq 0$$

These two requirements together imply that we must let $dx_2 = 0$. In contrast, we are free to choose dx_1. Thus, for instance, the vector $(dx_1, dx_2) = (2, 0)$ is an acceptable test vector, as is $(dx_1, dx_2) = (-1, 0)$. The latter test vector would plot in Fig. 13.2 as an arrow starting from (1, 0) and pointing in the due-west direction (not drawn), and it is clearly possible to draw a qualifying arc for it. (The curved boundary of the feasible region itself can serve as a qualifying arc.) On the other hand, the test vector $(dx_1, dx_2) = (2, 0)$ would plot as an arrow starting from (1, 0) and pointing in the due-east direction (not drawn). Since there is no way to draw a smooth arc tangent to this vector and lying entirely within the feasible region, no qualifying arcs exist for it. Hence the optimal solution point (1, 0) violates the constraint qualification.

Example 5 Referring to Example 2, let us illustrate that, after an additional constraint $2x_1 + x_2 \leq 2$ is added to Fig. 13.2, the point (1, 0) will satisfy the constraint qualification, thereby revalidating the Kuhn-Tucker conditions.

As in Example 4, we have to require $dx_2 \geq 0$ (because $x_2^* = 0$) and $dx_2 \leq 0$ (because the first constraint is exactly satisfied); thus, $dx_2 = 0$. But the second constraint is also exactly satisfied, thereby requiring

$$g_1^2\, dx_1 + g_2^2\, dx_2 = 2dx_1 + dx_2 = 2dx_1 \leq 0 \qquad \text{[by (13.23)]}$$

With nonpositive dx_1 and zero dx_2, the only admissible test vectors—aside from the null vector itself—are those pointing in the due-west direction in Fig. 13.2 from (1, 0). All of these lie along the horizontal axis in the feasible region, and it is certainly possible to draw a qualifying arc for each test vector. Hence, this time the constraint qualification indeed is satisfied.

Linear Constraints

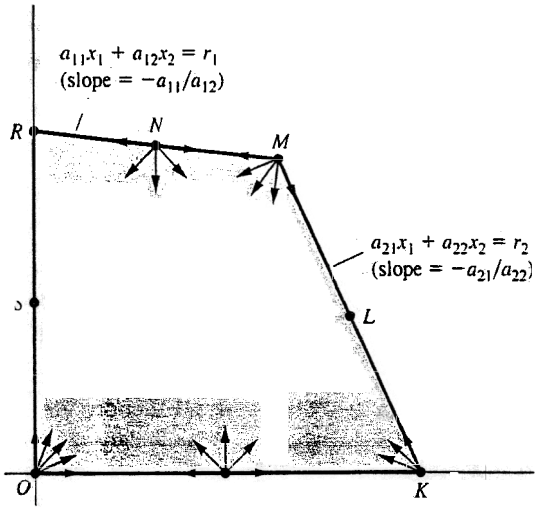
Earlier, in Example 3, it was demonstrated that the convexity of the feasible set does not guarantee the validity of the Kuhn-Tucker conditions as necessary conditions. However, if the feasible region is a convex set formed by *linear* constraints only, then the constraint qualification will invariably be met, and the Kuhn-Tucker conditions will always hold at an optimal solution. This being the case, we need never worry about boundary irregularities when dealing with a nonlinear programming problem with linear constraints.

Example 6 Let us illustrate the linear-constraint result in the two-variable, two-constraint framework. For a maximization problem, the linear constraints can be written as

$$a_{11}x_1 + a_{12}x_2 \leq r_1$$
$$a_{21}x_1 + a_{22}x_2 \leq r_2$$

where we shall take all the parameters to be positive. Then, as indicated in Fig. 13.4, the first constraint border will have a slope of $-a_{11}/a_{12} < 0$, and the second, a slope of $-a_{21}/a_{22} < 0$. The boundary points of the shaded feasible region fall into the following five types: (1) the point of origin, where the two axes intersect, (2) points that lie on one axis segment, such

FIGURE 13.4



as J and S, (3) points at the intersection of one axis and one constraint border, namely, K and R, (4) points lying on a single constraint border, such as L and N, (5) the point of intersection of the two constraints, M. We may briefly examine each type in turn with reference to the satisfaction of the constraint qualification.

1. At the origin, no constraint is exactly satisfied, so we may ignore (13.23). But since $x_1 = x_2 = 0$, we must choose test vectors with $dx_1 \geq 0$ and $dx_2 \geq 0$, by (13.22). Hence all test vectors from the origin must point in the due-east, due-north, or northeast directions, as depicted in Fig. 13.4. These vectors all happen to fall within the feasible set, and a qualifying arc clearly can be found for each.
2. At a point like J, we can again ignore (13.23). The fact that $x_2 = 0$ means that we must choose $dx_2 \geq 0$, but our choice of dx_1 is free. Hence all vectors would be acceptable except those pointing southward ($dx_2 < 0$). Again all such vectors fall within the feasible region, and there exists a qualifying arc for each. The analysis of point S is similar.
3. At points K and R, both (13.22) and (13.23) must be considered. Specifically, at K, we have to choose $dx_2 \geq 0$ since $x_2 = 0$, so that we must rule out all southward arrows. The second constraint being exactly satisfied, moreover, the test vectors for point K must satisfy

$$g_1^2\, dx_1 + g_2^2\, dx_2 = a_{21}\, dx_1 + a_{22}\, dx_2 \leq 0 \qquad (13.24)$$

   Since at K we also have $a_{21}x_1 + a_{22}x_2 = r_2$ (second constraint border), however, we may add this equality to (13.24) and modify the restriction on the test vector to the form

$$a_{21}(x_1 + dx_1) + a_{22}(x_2 + dx_2) \leq r_2 \qquad (13.24')$$

   Interpreting $(x_j + dx_j)$ to be the new value of x_j attained at the arrowhead of a test vector, we may construe (13.24') to mean that all test vectors must have their arrowheads located on or below the second constraint border. Consequently, all these vectors must again fall within the feasible region, and a qualifying arc can be found for each. The analysis of point R is analogous.
4. At points such as L and N, neither variable is zero and (13.22) can be ignored. However, for point N, (13.23) dictates that

$$g_1^1\, dx_1 + g_2^1\, dx_2 = a_{11}\, dx_1 + a_{12}\, dx_2 \leq 0 \qquad (13.25)$$

   Since point N satisfies $a_{11}\, dx_1 + a_{12}\, dx_2 = r_1$ (first constraint border), we may add this equality to (13.25) and write

$$a_{11}(x_1 + dx_1) + a_{12}(x_2 + dx_2) \leq r_1 \qquad (13.25')$$

   This would require the test vectors to have arrowheads located on or below the first constraint border in Fig. 13.4. Thus we obtain essentially the same kind of result encountered in the other cases. This analysis of point L is analogous.
5. At point M, we may again disregard (13.22), but this time (13.23) requires all test vectors to satisfy both (13.24) and (13.25). Since we may modify the latter conditions to the forms in (13.24') and (13.25'), all test vectors must now have their arrowheads located on or below the first as well as the second constraint borders. The result thus again duplicates those of the previous cases.

In this example, it so happens that, for every type of boundary point considered, the test vectors all lie within the feasible region. While this locational feature makes the qualifying arcs easy to find, it is by no means a prerequisite for their existence. In a problem with a

nonlinear constraint border, in particular, the constraint border itself may serve as a qualifying arc for some test vector that lies outside of the feasible region. An example of this can be found in one of the problems below.

EXERCISE 13.2

1. Check whether the solution point $(x_1^*, x_2^*) = (2, 6)$ in Example 3 satisfies the constraint qualification.
2. Maximize $\pi = x_1$
   subject to $x_1^2 + x_2^2 \leq 1$
   and $x_1, x_2 \geq 0$
   Solve graphically and check whether the optimal-solution point satisfies (*a*) the constraint qualification and (*b*) the Kuhn-Tucker conditions.
3. Minimize $C = x_1$
   subject to $x_1^2 - x_2 \geq 0$
   and $x_1, x_2 \geq 0$
   Solve graphically. Does the optimal solution occur at a cusp? Check whether the optimal solution satisfies (*a*) the constraint qualification and (*b*) the Kuhn-Tucker minimum conditions.
4. Minimize $C = x_1$
   subject to $-x_2 - (1 - x_1)^3 \geq 0$
   and $x_1, x_2 \geq 0$
   Show that (*a*) the optimal solution $(x_1^*, x_2^*) = (1, 0)$ does not satisfy the Kuhn-Tucker conditions, but (*b*) by introducing a new multiplier $\lambda_0 \geq 0$, and modifying the Lagrangian function (13.15) to the form

$$Z_0 = \lambda_0 f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{m} \lambda_i [r_i - g^i(x_1, x_2, \ldots, x_n)]$$

   the Kuhn-Tucker conditions can be satisfied at (1, 0). (*Note:* The Kuhn-Tucker conditions on the multipliers extend to only $\lambda_1, \ldots, \lambda_m$, but not to λ_0.)

Example

13.3 Economic Applications

War-Time Rationing

Typically during times of war the civilian population is subject to some form of rationing of basic consumer goods. Usually, the method of rationing is through the use of redeemable coupons used by the government. The government will supply each consumer with an allotment of coupons each month. In turn, the consumer will have to redeem a certain number of coupons at the time of purchase of a rationed good. This effectively means the consumer pays *two* prices at the time of the purchase. He or she pays both the coupon price and the monetary price of the rationed good. This requires the consumer to have both sufficient funds and sufficient coupons in order to buy a unit of the rationed good.

Consider the case of a two-good world where both goods, x and y, are rationed. Let the consumer's utility function be $U = U(x, y)$. The consumer has a fixed money budget of B